55^{th} Austrian Mathematical Olympiad

Junior Regional Competition—Solutions

11th June 2024

Problem 1. *Let x and y be positive real numbers with $x+y=1$. Prove that*

$$\frac{x+1}{y}+\frac{y+1}{x}\geq 6.$$

When does equality hold?

(Karl Czakler)

Answer. Equality holds exactly for $x=y=\frac{1}{2}$.

Solution. We have

$$\frac{x+1}{y}+\frac{y+1}{x}=\frac{x+x+y}{y}+\frac{y+x+y}{x}=2\left(\frac{x}{y}+\frac{y}{x}\right)+2.$$

For $x,y>0$, the AM-GM inequality gives

$$\frac{\frac{x}{y}+\frac{y}{x}}{2}\geq\sqrt{\frac{x}{y}\cdot\frac{y}{x}}=1,$$

which immediately implies the desired inequality.

Equality holds for $\frac{x}{y}=\frac{y}{x}$, i.e. $x=y=\frac{1}{2}$.

(Karl Czakler) □

Problem 2. *Let $ABCD$ be a trapezoid with parallel sides AB and CD, with $\angle BAD=90^\circ$ and with $AB+CD=BC$. Furthermore, let M be the mid-point of AD.*

Prove that $\angle CMB=90^\circ$.

(Karl Czakler)

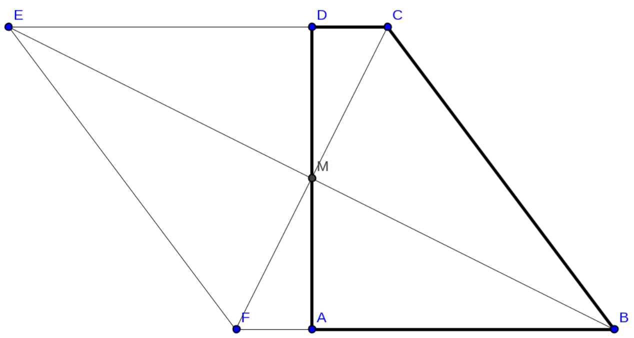

Figure 1: Problem 2

Solution. We reflect the points B and C in M and obtain the points E and F, respectively. We clearly have $EC=BF=AB+AF=AB+CD=BC=EF$, therefore, the quadrilateral $BCEF$ is a rhombus. Since the diagonals in a rhombus are orthogonal, we get $BE\perp CF$ and we obtain $\angle BMC=90^\circ$ as desired.

(Theresia Eisenkölbl) □

Problem 3. *Anna, Berta and Clara write the square numbers* $1, 4, 9, \ldots, 2025$ *on a blackboard, compute their sum and observe that it is divisible by* 3. *Then, they agree to the following game: In each round, Anna will cross out one number, then Berta will do the same, and then Clara will do the same. This continues until all numbers are crossed out. Clara has the goal that the sum of the remaining numbers after each round is divisible by* 3.

a) *Prove that Anna cannot stop Clara from reaching her goal if Clara has Berta's help.*

b) *Prove that Berta can stop Clara from reaching her goal even if Clara has Anna's help.*

(Richard Henner)

Solution. On the blackboard, we have 15 integers with residue 0 modulo 3 and 30 integers with residue 1 modulo 3. If, in a certain round, Berta and Clara cross out numbers that have the same residue modulo 3 as the number crossed out by Anna, then they have removed either $0+0+0$ or $1+1+1$ modulo 3.

In both cases, the sum does not change modulo 3 and therefore remains divisible by 3. Since 15 and 30 are multiples of 3, it is possible for Berta and Clara to always choose the same residue as Anna. Therefore, Anna cannot stop Clara from reaching her goal if Clara has Berta's help.

However, if Berta chooses in the first round a residue modulo 3 that is different from the one chosen by Anna, they have crossed out $0+1$ or $1+0$ modulo 3. Therefore, Clara's only choices for the sum of the remaining numbers after the first round are 1 or 2 modulo 3. In both cases, the sum is not divisible by 3. Therefore, Clara has failed in her goal already in the first round if Berta plays uncooperatively.

(Richard Henner) □

Problem 4. *Determine the maximal number of consecutive positive integers such that each of these integers has a common divisor with* 2024 *greater than* 1.

(Walther Janous)

Answer. The maximal number is 5.

Solution. We observe that $2024 = 2^3 \cdot 11 \cdot 23$. An integer has a common divisor greater than 1 with 2024 if and only if it is divisible by 2, 11 or 23.

Let N be the desired maximal number. Only each 11th integer is divisible by 11. That means that if z is divisible by 11, then $z+1$, $z+2$, $\ldots$, $z+10$ are not divisible by 11. Analogously, 23 divides only every 23rd integer. Six consecutive integers contain exactly three odd numbers. At most one of them is divisible by 11 and at most one of them is divisible by 23. This shows that $N \leq 5$.

Now, we try to find five consecutive integers n, $n+1$, $n+2$, $n+3$, $n+4$ that have a common divisor greater than 1 with 2024.

We can do that in the following way:

n	even
$n+1$	divisible by 11
$n+2$	even
$n+3$	divisible by 23
$n+4$	even

That means that we want $n+1 = 11k$ and $n+3 = 23l$ with k and l odd. If we subtract the second equation from the first, we get

$$\begin{aligned} 2 &= 23l - 11k \\ &= l + 11(2l - k). \end{aligned}$$

We obtain $l \equiv 2 \pmod{11}$. We see that $l = 13$ works, since we get $n+3 = 23l = 299$ and therefore $n = 296$ and the five consecutive integers 296, 297, 298, 299, 300, which have the desired property.

Therefore, $N = 5$.

(Walther Janous) □