53^{rd} Austrian Mathematical Olympiad

Junior Regional Competition—Solutions

14th June 2022

Problem 1. *Show that for all real numbers x and y with $x > -1$, $y > -1$ and $x + y = 1$ the inequality*

$$\frac{x}{y+1} + \frac{y}{x+1} \geq \frac{2}{3}$$

holds.

When does equality hold?

(Walther Janous)

Solution. We replace the 1s on the left-hand side with $x + y$ to homogenize the inequality and obtain

$$\frac{x}{x+2y} + \frac{y}{2x+y} \geq \frac{2}{3}.$$

The denominators are positive because $x + 1$ and $y + 1$ were positive and we just rewrote them, so we can multiply with all denominators and obtain

$$\begin{aligned}
& 3x(2x+y) + 3y(x+2y) \geq 2(x+2y)(2x+y) \\
\iff \quad & 6x^2 + 3xy + 3xy + 6y^2 \geq 4x^2 + 4y^2 + 10xy \\
\iff \quad & 2x^2 - 4xy + 2y^2 \geq 0 \\
\iff \quad & x^2 - 2xy + y^2 \geq 0 \\
\iff \quad & (x-y)^2 \geq 0
\end{aligned}$$

This is clearly true and equality holds for $x = y$ which means $x = y = \frac{1}{2}$ due to the condition $x + y = 1$.

(Theresia Eisenkölbl) □

Problem 2. *Consider a 13×2 rectangular board and an arbitrarily large number of dominoes in sizes 2×1 and 3×1.*

We want to cover the board with dominoes without gaps or overlaps or parts of a domino outside the board. Additionally, all dominoes have to have the same orientation, i. e. all their longer sides have to be parallel.

How many configurations are possible?

(Walther Janous)

Answer. There are 257 configurations.

Solution. We will assume that the board is placed with the longer side horizontally.

If the longer sides of the dominoes are parallel to the shorter side of the board, we can only use 2×1 dominoes and all of them are placed vertically, which gives exactly one configuration.

If the longer sides of the dominoes are parallel to the longer side of the board, we have to place all the dominoes horizontally. For a certain row of the board, let x and y be the number of dominoes of size 2×1 and 3×1, respectively. Then we must have $2x + 3y = 13$.

Clearly, y has to be odd, positive and less than 5 which gives the two pairs of solutions $x = 2$, $y = 3$ and $x = 5$, $y = 1$.

In the first case, we can still decide were to place the 2 ordinary dominoes among the 5 dominoes. There are $\binom{5}{2} = 10$ possibilities to choose the 2 places among the 5. Similarly, in the second case, there are $\binom{6}{1} = 6$ to choose the place of the single 3×1 among the 6 dominoes.

This means that there 16 possibilites for each of the two rows in total. Since we can choose them independently, we obtain $16^2 = 256$ possibilities.

In total, we have found $1 + 256 = 257$ configurations.

(Reinhard Razen) □

Problem 3. *Consider the semicircle with center M and diameter AB. Let P be a point on the semicircle different from A and B and let Q be the midpoint of the arc AP. Let S be the intersection of the line BP with the parallel of PQ through M.*

Prove that $PM = PS$.

(Karl Czakler)

Solution. We first note that

$$\angle PSM = 180^\circ - \angle QPB$$

because of the equal angles of PS with the two parallel lines MS and PQ.

The inscribed angle theorem for the four points A, B, P and Q immediately gives

$$180^\circ - \angle QPB = \angle BAQ = \angle MAQ.$$

On the other hand,

$$\angle PMS = \angle MPQ,$$

because of the equal angles of MP with the two given parallel lines.

However, we clearly have

$$\angle MPQ = \angle MAQ,$$

because the points A and P are symmetric to each other with respect to MQ, which gives $\angle PSM = \angle PMS$, as desired.

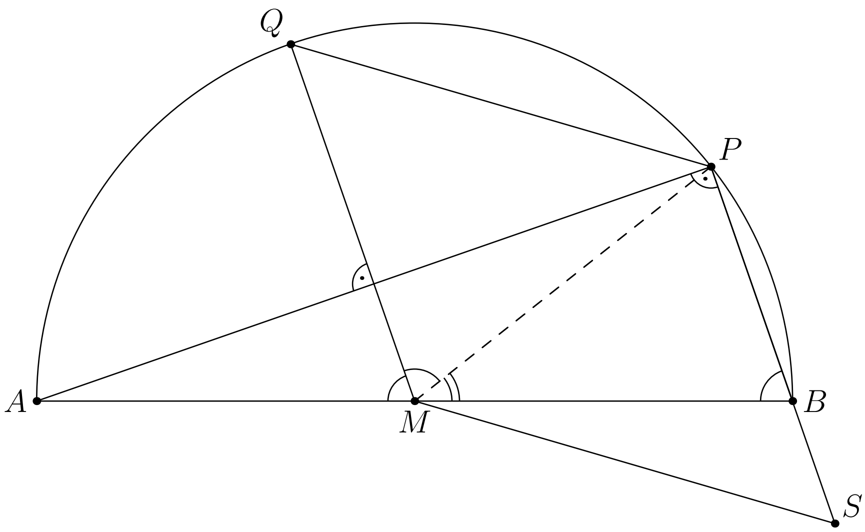

Figure 1: Problem 3

(Theresia Eisenkölbl) □

Problem 4. *Find all primes p, q and r with $p + q^2 = r^4$.*

(Karl Czakler)

Answer. The only solution is $p = 7$, $q = 3$, $r = 2$.

Solution. The equation can be written in the form

$$p = r^4 - q^2 = (r^2 - q)(r^2 + q).$$

Since p is prime and the second factor positive and larger than the other factor, we obtain $r^2 - q = 1$ and $r^2 + q = p$.

From $r^2 - q = 1$, we get $q = r^2 - 1 = (r - 1)(r + 1)$. Since q is prime and the second factor positive and larger than the other factor, we obtain $r - 1 = 1$ and therefore, $r = 2$ and $q = 3$. With $r^2 + q = p$, we get $p = 7$, and the triple $(7, 3, 2)$ is indeed a solution of the given equation.

(Karl Czakler) □