53rd Austrian Mathematical Olympiad

National Competition—Final Round (Day 1)

25th May 2022

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1. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with

$$a - f(b) \mid af(a) - bf(b) \text{ for all } a, b \in \mathbb{Z}_{>0}.$$

(Theresia Eisenkölbl)

2. Let ABC be an acute, scalene triangle with orthocenter H, and let M be the midpoint of segment AB, and w the angular bisector of angle $\angle ACB$. Let S be the intersection of w and the perpendicular bisector of AB, and F the foot of the altitude from H onto w.

   Prove that segments MS and MF are of equal length.

(Karl Czakler)

3. Lisa writes a positive integer in the decimal system on a board and repeats the following steps:

   The last digit is deleted from the number on the board and then four times the deleted digit is added to the remaining shorter number (or to 0 if the original number was a single digit). The result of this calculation is now the new number on the board.

   This is repeated until the first time she gets a number that has already been on the board.

   (a) Show that the sequence of steps always terminates.

   (b) What is the last number on the board if Lisa starts with the number $53^{2022} - 1$?

   Example: If Lisa starts with the number 2022, she gets $202 + 4 \times 2 = 210$ in the first step and then subsequently

$$2022 \mapsto 210 \mapsto 21 \mapsto 6 \mapsto 24 \mapsto 18 \mapsto 33 \mapsto 15 \mapsto 21.$$

   Since Lisa gets 21 a second time, she stops.

(Stephan Pfannerer)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

53rd Austrian Mathematical Olympiad

National Competition—Final Round (Day 2)

26th May 2022

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4. Decide if for every polynomial P of degree ≥ 1 with integer coefficients, there are infinitely many primes that each divide a $P(n)$ for a positive integer n.

   (Walther Janous)

5. Let ABC be an isosceles triangle with base AB.

   We choose an interior point P of the altitude in C. The circle with diameter CP intersects the line connecting B and P a second time in D_P and the line connecting points A and C a second time in E_P.

   Prove that there exists a point F, such that for every choice of P the points D_P, E_P and F are collinear.

   (Walther Janous)

6. (a) Prove that a square with sidelength 1000 can be tiled with 31 squares such that at least one of them has sidelength smaller than 1.

   (b) Prove that there is also a tiling with 30 squares with the same properties.

   (Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.