

53rd Austrian Mathematical Olympiad
National Competition—Final Round (Day 1)
25th May 2022

1. Find all functions $f: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ with

$$a - f(b) \mid af(a) - bf(b) \text{ for all } a, b \in \mathbb{Z}_{>0}.$$

(Theresia Eisenkölbl)

2. Let ABC be an acute, scalene triangle with orthocenter H , and let M be the midpoint of segment AB , and w the angular bisector of angle $\angle ACB$. Let S be the intersection of w and the perpendicular bisector of AB , and F the foot of the altitude from H onto w . Prove that segments MS and MF are of equal length.

(Karl Czakler)

3. Lisa writes a positive integer in the decimal system on a board and repeats the following steps:

The last digit is deleted from the number on the board and then four times the deleted digit is added to the remaining shorter number (or to 0 if the original number was a single digit). The result of this calculation is now the new number on the board.

This is repeated until the first time she gets a number that has already been on the board.

- (a) Show that the sequence of steps always terminates.
(b) What is the last number on the board if Lisa starts with the number $53^{2022} - 1$?

Example: If Lisa starts with the number 2022, she gets $202 + 4 \times 2 = 210$ in the first step and then subsequently

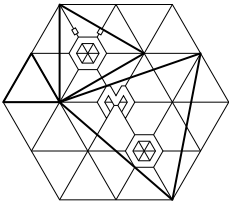
$$2022 \mapsto 210 \mapsto 21 \mapsto 6 \mapsto 24 \mapsto 18 \mapsto 33 \mapsto 15 \mapsto 21.$$

Since Lisa gets 21 a second time, she stops.

(Stephan Pfannerer)

Working time: $4\frac{1}{2}$ hours.

Each problem is worth 8 points.



53rd Austrian Mathematical Olympiad
National Competition—Final Round (Day 2)
26th May 2022

4. Decide if for every polynomial P of degree ≥ 1 with integer coefficients, there are infinitely many primes that each divide a $P(n)$ for a positive integer n .

(Walther Janous)

5. Let ABC be an isosceles triangle with base AB .

We choose an interior point P of the altitude in C . The circle with diameter CP intersects the line connecting B and P a second time in D_P and the line connecting points A and C a second time in E_P .

Prove that there exists a point F , such that for every choice of P the points D_P , E_P and F are collinear.

(Walther Janous)

6. (a) Prove that a square with sidelength 1000 can be tiled with 31 squares such that at least one of them has sidelength smaller than 1.

(b) Prove that there is also a tiling with 30 squares with the same properties.

(Walther Janous)

Working time: $4\frac{1}{2}$ hours.

Each problem is worth 8 points.