49^{th} Austrian Mathematical Olympiad

Regional Competition (Qualifying Round)

5th April 2018

1. Let a and b be nonnegative real numbers satisfying $a + b < 2$.

   Prove the inequality
$$\frac{1}{1+a^2} + \frac{1}{1+b^2} \le \frac{2}{1+ab}$$
   and determine all a and b yielding equality.

   (Gottfried Perz)

2. Let k be a circle with radius r and AB a chord of k such that $AB > r$. Furthermore, let S be the point on the chord AB satisfying $AS = r$. The perpendicular bisector of BS intersects k in the points C and D. The line through D and S intersects k for a second time in point E.

   Show that the triangle CSE is equilateral.

   (Stefan Leopoldseder)

3. Let $n \ge 3$ be a natural number.

   Determine the number a_n of all subsets of $\{1, 2, \ldots, n\}$ consisting of three elements such that one of them is the arithmetic mean of the other two.

   (Walther Janous)

4. Let $d(n)$ be the number of all positive divisors of a natural number $n \ge 2$.

   Determine all natural numbers $n \ge 3$ such that
$$d(n-1) + d(n) + d(n+1) \le 8.$$

   (Richard Henner)

Working time: 4 hours.
Each problem is worth 8 points.