48^{th} Austrian Mathematical Olympiad

Beginners' Competition—Solutions

13th June 2017

Problem 1. *The nonnegative real numbers a and b satisfy $a+b=1$. Prove that*

$$\frac{1}{2} \le \frac{a^3+b^3}{a^2+b^2} \le 1.$$

When do we have equality in the right inequality and when in the left inequality?

(Walther Janous)

Solution. (Gerhard Kirchner) By algebraic manipulation we achieve

$$\frac{a^3+b^3}{a^2+b^2} = (a+b)\frac{a^2-ab+b^2}{a^2+b^2} = 1 - \frac{ab}{a^2+b^2}.$$

From this the right inequality is evident with equality for $ab=0$, i. e. for $a=0$, $b=1$ and for $a=1$, $b=0$. The left inequality is equivalent to

$$\frac{1}{2} \le 1 - \frac{ab}{a^2+b^2} \iff \frac{ab}{a^2+b^2} \le \frac{1}{2} \iff 2ab \le a^2+b^2 \iff 0 \le (a-b)^2.$$

This inequality is obvious with equality for $a=\frac{1}{2}$. In this case also $b=\frac{1}{2}$. □

Problem 2. *In the isosceles triangle ABC with $\overline{AC}=\overline{BC}$ we denote by D the foot of the altitude through C. The midpoint of CD is denoted by M. The line BM intersects AC in E. Prove that the length of AC is three times that of CE.*

(Erich Windischbacher)

Solution. (Gerhard Kirchner) We consider the centroid S of triangle DBC, which lies on the axis BM; see Figure 1. The centroidal axis DS bisects the segment BC, thus DS is parallel to AC by the intercept

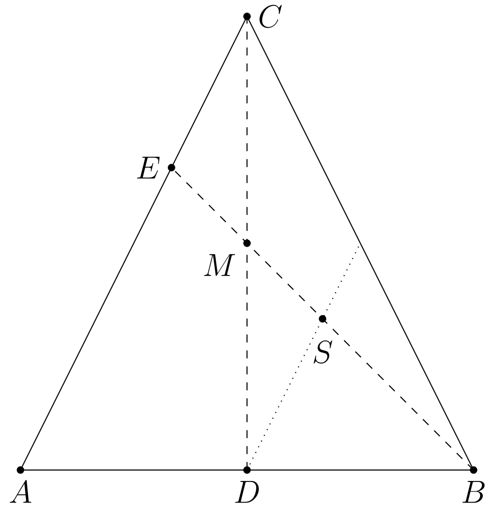

Figure 1: Problem 2

theorem. Since the centroid divides the centroidal axis in the ratio $2:1$, we have the same division ratio on the parallel line AC, i. e. E divides AC in the ratio $2:1$. □

Problem 3. *Anthony writes down in order all positive integers which are divisible by* 2*. Bertha writes down in order all positive integers which are divisible by* 3*. Claire writes down in order all positive integers which are divisible by* 4*. Orderly Dora writes all numbers written by the other three. Thereby she puts them in order by size and does not repeat a number. What is the 2017th number in her list?*

(Richard Henner)

Solution. (Richard Henner) Dora can ignore Claire's numbers, since Anthony has already written them all. Considering the numbers up to 3000 we see that Anthony has already written down 1500 of them and Bertha has written 1000 of them, 500 of which have been written twice, which are ignored by Dora in their second occurrence. Hence Dora denotes exactly 2000 numbers up to 3000. The next 17 numbers written by Dora are $3002, 3003, 3004, 3006, 3008, 3009, 3010, 3012, 3014, 3015, 3016, 3018, 3020, 3021, 3022, 3024, 3026$. Thus 3026 is the 2017^{th} number on Dora's list. □

Problem 4. *How many solutions does the equation*

$$\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{17} \right\rfloor$$

have over the set of positive integers?
Therein $\lfloor a \rfloor$ denotes the largest integer that is less than or equal to a.

(Karl Czakler)

Solution. (Gerhard Kirchner) Applying Euclidean division of x by 17 and 20 resp. gives

$$x = 20a + b = 17c + d, \qquad a, b, c, d \in \mathbb{N}, \quad 0 \le b \le 19, \quad 0 \le d \le 16.$$

The given equation then states $a = c$ and we obtain $3a = d - b$. Hence we have to find the number of possibilities for $b \in \{0, 1, \ldots, 19\}$ and $d \in \{0, 1, \ldots, 16\}$ such that $d \ge b$ und $3 \mid d - b$. Moreover we need to have $x > 0$, i. e. $b = d = 0$ is not allowed.
For each possible value of d we list the number of possible numbers b in the same residue class mod 3:

d	0	1, 2	3, 4, 5	6, 7, 8	9, 10, 11	12, 13, 14	15, 16
number of possible b for each d	–	1	2	3	4	5	6

We therefore have $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 3 + 4 \cdot 3 + 5 \cdot 3 + 6 \cdot 2 = 56$ solutions. □