51^{st} Austrian Mathematical Olympiad

National Competition—Preliminary Round

21st May 2020

1. Let x, y and z be positive real numbers subject to $x \geq y + z$.

   Prove the inequality
   $$\frac{x+y}{z} + \frac{y+z}{x} + \frac{z+x}{y} \geq 7.$$
   When does equality hold?

   (Walther Janous)

2. Let ABC be a right triangle with its right angle in C, and circumcenter U. Points D and E lie on the sides AC and BC, respectively, such that $\angle EUD = 90°$ holds. Furthermore, let F and G denote the feet of D and E on AB, respectively.

   Prove that FG is half as long as AB.

   (Walther Janous)

3. Three positive integers are written on a blackboard. In each move, the numbers are first assigned the labels a, b and c in a way that $a > \gcd(b, c)$ holds, and then a is replaced with $a - \gcd(b, c)$. The game ends if there is no possible labelling with the desired property.

   Show that the game always ends and always reaches the same three numbers $x \leq y \leq z$ for the same starting numbers.

   (Theresia Eisenkölbl)

4. Determine all positive integers N such that $2^N - 2N$ is the square of an integer.

   (Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.