54^{th} Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

29th April 2023

Problem 1. *Let a, b, c, d be real numbers with $0 < a, b, c, d < 1$ and $a + b + c + d = 2$. Show that*

$$\sqrt{(1-a)(1-b)(1-c)(1-d)} \leq \frac{ac+bd}{2}.$$

Are there infinitely many cases of equality?

(Josef Greilhuber)

Solution. Squaring the given inequality and multiplying by 16, we get

$$(2-2a)(2-2b)(2-2c)(2-2d) \leq 4(ac+bd)^2.$$

We homogenize by replacing the first 2 in each parenthesis on the left side by $a + b + c + d$ and get the homogeneous inequality

$$(b+d-(a-c))(a+c-(b-d))(b+d+a-c)(a+c+b-d) \leq 4(ac+bd)^2.$$

We evaluate the left-hand side by repeatedly combining two factors and get

$$\begin{aligned}
&(b+d-(a-c))\,(a+c-(b-d))\,(b+d+a-c)\,(a+c+b-d) \\
&= \big((a+c)^2-(b-d)^2\big)\big((b+d)^2-(a-c)^2\big) \\
&= \left(2ac+2bd+a^2+c^2-b^2-d^2\right)\left(2ac+2bd-a^2-c^2+b^2+d^2\right) \\
&= 4\,(ac+bd)^2-(a^2+c^2-b^2-d^2)^2 \leq 4\,(ac+bd)^2,
\end{aligned}$$

which proves the inequality.

Equality holds for $a^2 + c^2 = b^2 + d^2$, in particular for $a = b$ and $c = d = 1 - a$ with $0 < a < 1$. Therefore, there are infinitely many equality cases.

(Josef Greilhuber) □

Problem 2. *Let ABC be a triangle. Let P be the point on the extension of BC beyond B such that $BP = BA$. Let Q be the point on the extension of BC beyond C such that $CQ = CA$.*

Prove that the circumcenter O of the triangle APQ lies on the angle bisector of the angle $\angle BAC$.

(Karl Czakler)

Solution. Since ACQ is an isosceles triangle, the perpendicular bisector of AQ is the angle bisector of $\angle QCA$. But the perpendicular bisector of AQ also passes through the circumcenter O of the triangle APQ.

Therefore, O lies on the angle bisector of $\angle QCA$ which is the exterior angle bisector of $\angle ACB$ by definition of Q.

Analogously, the point O lies also on the exterior angle bisector of $\angle CBA$. Therefore, the point O is the intersection of the two exterior angle bisectors which makes it the excenter of the excircle of ABC tangent to BC. This excenter lies on the angle bisector of $\angle BAC$ as desired.

(Theresia Eisenkölbl) □

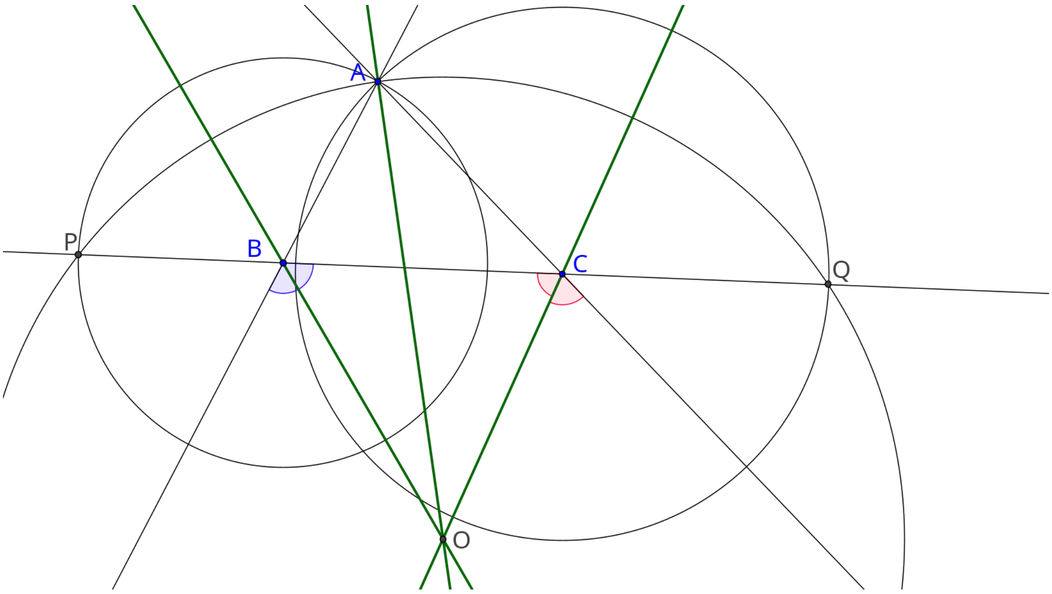

Figure 1: Problem 2

Problem 3. *Let n be a positive integer. What proportion of the non-empty subsets of $\{1, 2, \ldots, 2n\}$ has a smallest element that is odd?*

(Birgit Vera Schmidt)

Solution. The number of subsets of $\{1, 2, \ldots, 2n\}$ that have k as smallest element is 2^{2n-k} for $1 \leq k \leq 2n$ since each element bigger than k is either contained in the subset or not.

The number O of subsets with an odd smallest element is therefore equal to

$$O = 2^{2n-1} + 2^{2n-3} + \cdots + 2^3 + 2^1 = 2 \cdot (4^{n-1} + 4^{n-2} + \cdots + 4^1 + 4^0).$$

The number E of subsets with an even smallest element is equal to

$$E = 2^{2n-2} + 2^{2n-4} + \cdots + 2^2 + 2^0 = 4^{n-1} + 4^{n-2} + \cdots + 4^1 + 4^0.$$

This implies $O = 2E$ and consequently the desired proportion is $2/3$.

(Birgit Vera Schmidt) □

Problem 4. *Determine all pairs of positive integers (n, k) for which*

$$n! + n = n^k$$

holds.

(Michael Reitmeir)

Answer. The only solutions are $(2, 2), (3, 2)$ and $(5, 3)$.

Solution. Because of $n! + n > n$, we immediately get $k \geq 2$. We divide both sides of the equation by n and get

$$(n-1)! + 1 = n^{k-1}.$$

Now, we distinguish two cases:

- n is not a prime.
  Since n is clearly not 1, we can write n as $n = ab$ for integers a, b with $1 < a, b < n$ which implies $1 < a \leq n-1$ and therefore $a \mid (n-1)!$. We conclude that $a > 1$ is relatively prime to the left-hand side $(n-1)! + 1$, but a divides the right-hand side n^{k-1}. This is not possible, so there are no solutions in this case.

- n is a prime.
  We check $n = 2, 3, 5$ and find the solutions $(2, 2), (3, 2)$ and $(5, 3)$.

  From now on, let $n \geq 7$. We get

$$
\begin{aligned}
& (n-1)! = n^{k-1} - 1 \\
\implies \quad & (n-1)! = (1 + n + n^2 + \cdots + n^{k-2})(n-1) \\
\implies \quad & (n-2)! = 1 + n + n^2 + \cdots + n^{k-2}
\end{aligned}
$$

  Since n is prime and bigger than 3, the number $n-1$ is even and not a prime. Furthermore, $n-1$ is not the square of a prime since 4 is the only even square of a prime and $n - 1 \geq 6$. Therefore, we get $n - 1 = ab$ with $1 < a, b \leq n - 1$ and $a \neq b$. We obtain that $(n-2)!$ contains the separate factors a and b and is therefore divisible by $ab = n - 1$ which implies $(n-2)! \equiv 0 \bmod (n-1)$. Furthermore, $n \equiv 1 \bmod (n-1)$, and therefore

$$0 \equiv 1 + 1 + 1^2 + \cdots + 1^{k-2} \equiv k - 1 \bmod (n-1).$$

  We conclude that $n-1$ divides $k-1$ and we write $k - 1 = l(n-1)$ for a positive integer l. The case $k = 1$ and $l = 0$ has already been treated. Therefore, we get $k - 1 \geq n - 1$.

  However,

$$(n-1)! = 1 \cdot 2 \cdot 3 \cdots (n-1) < \underbrace{(n-1) \cdot (n-1) \cdots (n-1)}_{n-1 \text{ times}} = (n-1)^{n-1},$$

  and therefore

$$n^{k-1} = (n-1)! + 1 \leq (n-1)^{n-1} < n^{n-1} \leq n^{k-1},$$

  giving a contradiction. So there are no further solutions.

(Michael Reitmeir) □