



# 40th Austrian Mathematical Olympiad

## Regional Competition for Advanced Students

April 23, 2009

1. State a domain  $M \subseteq \mathbb{R}^+$  as large as possible such that for all  $a, b, c, d \in M$  the inequality

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+b} + \sqrt{c+d}$$

holds. Does

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+c} + \sqrt{b+d}$$

also hold for all  $a, b, c, d \in M$ ? (Remark:  $\mathbb{R}^+$  is the set of all positive real numbers.)

2. How many integer solutions  $(x_0, x_1, x_2, x_3, x_4, x_5, x_6)$  does the equation

$$2x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 9$$

have?

3. Given is an acute-angled triangle  $ABC$  (counterclockwise labeled) with altitude base points  $D$  (on  $BC$ ),  $E$  (on  $AC$ ) and  $F$  (on  $AB$ ). Further let  $P$ ,  $Q$  and  $R$  be defined as follows:

- $P$  is the base point of the altitude of  $F$  on  $BC$  in the triangle  $CFB$ .
- $P$  is the base point of the altitude of  $D$  on  $AC$  in the triangle  $ADC$ .
- $P$  is the base point of the altitude of  $E$  on  $AB$  in the triangle  $AEB$ .

The six points  $D$ ,  $E$ ,  $F$ ,  $P$ ,  $Q$  and  $R$  form with suitable numbering  $T_1T_2T_3T_4T_5T_6$  (counterclockwise with  $T_1 = P$ ) a convex hexagon (all angles smaller than 180 degrees).

Show: In this convex hexagon there is no point which lies on all three diagonals  $T_1T_4$ ,  $T_3T_6$  and  $T_5T_2$ .

4. Two distinct arithmetic sequences  $\langle a_0, a_1, \dots, a_n = a_0 + nd, \dots \rangle$  are essentially different, if they do not differ only by the absence of finitely many members at the beginning of one of them. How many pairwise essentially different non-constant arithmetic sequences of positive integers are there which contain an infinite non-constant geometric sequence whose third member is  $40 \cdot 2009 = 80360$ ?