36th Austrian Mathematical Olympiad

Regional Competition for Advanced Students

May 3, 2005

1. Show that for all natural numbers $n \geq 2005$ the inequalities

$$(n+830)^{2005} < n(n+1)\dots(n+2004) < (n+1002)^{2005}$$

hold.

2. A semicircle h with diameter AB and center M is drawn. A second semicircle k with diameter MB is drawn on the same side of the line AB. Let X and Y be points on k such that the arc BX is one and a half times as long as the arc BY. The line MY intersects the line BX in D and the larger semicircle h in C.
Show that Y is the midpoint of the line segment CD.

3. For which real values of k and d does the system

$$\begin{aligned} x^3 + y^3 &= 2 \\ y &= kx + d \end{aligned}$$

have no real solutions (x, y)?

4. Show that if an infinite arithmetic sequence $(a_n = a_0 + nd)$ of positive real numbers contains two different powers of an integer $a > 1$, then it also contains an infinite geometric sequence $(b_n = b_0 q^n)$ of real numbers.