41st Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 13, 2010

1. Let $f(n) = \sum_{k=0}^{2010} n^k = 1 + n + n^2 + \cdots + n^{2010}$.

   Show that the following holds for each non-negative integer m with $2 \leq m \leq 2010$:
   There is no non-negative integer n, such that m divides $f(n)$.

2. For each positive integer n define the function $f_n(x) = \sum_{k=1}^{n} |x - k|$ for all real numbers x.

   For each two-digit number n (in decimal notation) determine the set of solutions L_n of the inequality $f_n(x) < 41$.

3. Let $M_n = \{0, 1, 2, \ldots, n\}$ be the set of non-negative integers that are less or equal to n. A subset S of M_n is called *distinguished* if it is non-empty and for each integer $k \in S$ there exists a k-element subset T_k of S.

   Determine the number $a(n)$ of distinguished subsets of M_n.

4. In a triangle ABC, if one draws through some interior point P the three parallels to the sides, then the triangle ABC is decomposed into three quadrangles (in the corners) and three triangles which rest on the sides.

   (a) Show that if $P = I$ is the center of the incircle, then the circumference of each of the new small triangles is equal to the length of the side on which it rests.

   (b) Determine for a given triangle ABC all interior points P for which the circumference of each of the new small triangles is equal to the length of the side on which it rests.

   (c) For which interior point P is the sum of the areas of the three triangles minimal?

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Part 2, Day 1, June 2, 2010

1. Show that

$$\frac{(x-y)^7+(y-z)^7+(z-x)^7-(x-y)(y-z)(z-x)\left((x-y)^4+(y-z)^4+(z-x)^4\right)}{(x-y)^5+(y-z)^5+(z-x)^5} \geq 3$$

   holds for all triples of distinct integers x, y, z. When does equality hold?

2. Determine all triples (x, y, z) of positive integers $x > y > z > 0$, such that $x^2 = y \cdot 2^z + 1$.

3. On a circular billiard table a ball rebounds from the rails as if the rail was the tangent to the circle at the point of impact.

   A regular hexagon with its vertices on the circle is drawn on a circular billiard table.

   A (point-shaped) ball is placed somewhere on the circumference of the hexagon, but not on one of its edges. Describe a periodical track of this ball with exactly four points at the rails. With how many different directions of impact can the ball be brought onto such a track?

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Part 2, Day 2, June 3, 2010

4. Consider the part of a lattice given by the corners $(0, 0)$, $(n, 0)$, $(n, 2)$ and $(0, 2)$. From a lattice point (a, b) one can move to $(a + 1, b)$ or to $(a + 1, b + 1)$ or tu $(a, b - 1)$, provided that the second point is also contained in the part of the lattice.

   How many ways are there to move from $(0, 0)$ to $(n, 2)$ considering these rules?

5. Two decompositions of a square into three rectangles are called substantially different, if reordering the rectangles does not change one into the other.

   How many substantially different decompositions of a 2010×2010 square in three rectangles with integer side lengths are there such that the area of one rectangle is equal to the arithmetic mean of the areas of the other rectangles?

6. A diagonal of a convex hexagon is called "long" if it decomposes the hexagon into two quadrangles.
   Each pair of long diagonals decomposes the hexagon into two triangles and two quadrangles.

   Given is a hexagon with the property, that for each decomposition by two long diagonals the resulting triangles are both isosceles with the side of the hexagon as base.

   Show that the hexagon has a circumcircle.