



# 40th Austrian Mathematical Olympiad

## Federal Competition for Advanced Students

### Part 1, May 17, 2009

1. Show, that the following inequality holds for all positive integers  $n$ :

$$3^{n^2} > (n!)^4$$

2. We generalize the functions factorial  $n! = n(n-1)(n-2)\cdots 2 \cdot 1$  and double factorial  $n!! = n(n-2)(n-4)\cdots 1$  for odd  $n$  and  $n!! = n(n-2)(n-4)\cdots 2$  for even  $n$ . For  $n > 0$ , we define the  $k$ -multifactorial  $F_k(n) = n(n-k)(n-2k)\cdots r$  with  $1 \leq r \leq k$  and  $n \equiv r \pmod{k}$ . Further,  $F_k(0) = 1$ .

Determine all non-negative integers  $n$ , such that  $F_{20}(n) + 2009$  is a square number (the square of an integer).

3. Given is a circuit with  $n$  stations ( $n > 1$ ) which can be traveled on in both directions. Each segment between two neighboring stations is called section. One of the stations is called Raach [the town where the ÖMO is held, ed.].

A bus should start in Raach and, after passing through  $n + 2$  sections, return to Raach. In doing so it should visit **each station at least once**.

For all  $n > 1$ , determine the number  $f(n)$  of tours which satisfy these conditions.

4. Let  $D$ ,  $E$  and  $F$  be the midpoints of the sides of the triangle  $ABC$  ( $D$  on  $BC$ ,  $E$  on  $CA$  and  $F$  on  $AB$ ).

Further let  $H_aH_bH_c$  be the triangle formed by the base points of the altitudes of the triangle  $ABC$ .

Let  $P$ ,  $Q$  and  $R$  be the midpoints of the sides of the triangle  $H_aH_bH_c$  ( $P$  on  $H_bH_c$ ,  $Q$  on  $H_cH_a$  and  $R$  on  $H_aH_b$ ).

Show: The lines  $PD$ ,  $QE$  and  $RF$  share a common point.



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Part 2, Day 1, June 10, 2009

1.  $A_{km}(x)$  is an exponential tower made up of  $k$  twos at the bottom, one  $x$  and, finally,  $m$  more twos on top.  $B_k(y)$  is also an exponential tower, made up of  $k$  fours at the bottom and one  $y$  on top. In other words,

$$A_{km}(x) = 2^{2^{2^{\dots^{2^x}}}} \quad \text{and} \quad B_k(y) = 4^{4^{\dots^y}}.$$

Determine all pairs  $(x, y)$  of non-negative integers, dependent on  $k > 0$ , such that  $A_{km}(x) = B_k(y)$ .

Comment: An exponential tower  $a^{b^c}$  is calculated as  $a^{(b^c)}$ .

2. (a) For positive integers  $a < b$  let

$$M(a, b) = \frac{\sum_{k=a}^b \sqrt{k^2 + 3k + 3}}{b - a + 1}$$

denote the arithmetic mean of the expressions  $\sqrt{k^2 + 3k + 3}$  for  $a \leq k \leq b$ . Calculate  $K(a, b) = [M(a, b)]$ .

- (b) Calculate

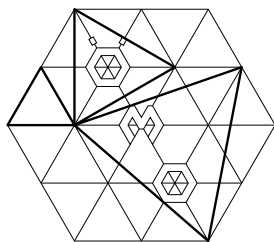
$$N(a, b) = \frac{\sum_{k=a}^b [\sqrt{k^2 + 3k + 3}]}{b - a + 1},$$

the arithmetic mean of the expressions  $[\sqrt{k^2 + 3k + 3}]$  for  $a \leq k \leq b$ .

(We assume that  $[x]$  is the largest integer less than or equal to  $x$ .)

3. We are given a triangle  $ABC$ . Determine all points  $P$  in the interior of the triangle, such that the following holds:

Let  $D$  be the common point of the continuation of  $AP$  with  $BC$  and let  $A'$  be the point on this continuation, for which  $\overline{AD} = \overline{DA'}$  holds. The triangles  $ABC$  and  $A'BC$  are congruent. (The points need not correspond in this order.) If  $B'$  and  $C'$  are defined in an analogous manner, triangles  $AB'C$  and  $ABC'$  are also congruent to  $ABC$ .



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Part 2, Day 2, June 11, 2009

4. For each positive integer  $a$  we consider the sequence  $\langle a_n \rangle$  with  $a_0 = a$  and  $a_n = a_{n-1} + 40^{n!}$  for  $n > 0$ .

Prove that every such sequence contains infinitely many numbers that are divisible by 2009.

5. Let  $n > 1$  and for  $1 \leq k \leq n$  let  $p_k = p_k(a_1, a_2, \dots, a_n)$  be the sum of the products of all possible combinations of  $k$  of the numbers  $a_1, a_2, \dots, a_n$ .

Furthermore, let  $P = P(a_1, a_2, \dots, a_n)$  be the sum of all  $p_k$  with odd values of  $k$  less than or equal to  $n$ .

How many different values are taken by the  $a_j$ , if all  $a_j$  ( $1 \leq j \leq n$ ) and  $P$  are prime?

6. Let  $ABCD$  be a quadrilateral and let  $P, Q, R$  and  $S$  be the mid-points of the sides  $AB, BC, CD$  and  $DA$  respectively. We call the quadrilateral  $PQRS$  *mid-point quadrilateral* of  $ABCD$ .

Determine all circumscribed quadrilaterals whose mid-point quadrilaterals are squares.