38th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 18, 2007

1. In a table with 2007 rows and 2007 columns and odd integer is written in each cell.

   For $1 \leq i \leq 2007$ let Z_i be the sum of the numbers in the i-th row and for $1 \leq j \leq 2007$ let S_j be the sum of the numbers in the j-th column.

   Further let A be the product of all Z_i and B the product of all S_j.

   Show that $A + B$ is different from zero.

2. For each positive integer n determine the largest possible value $C(n)$ such that for all n-tuples $(a_1, a_2, \ldots, a_n)$ of mutually different integers the inequality

$$(n+1)\sum_{j=1}^{n} a_j^2 - (\sum_{j=1}^{n} a_j)^2 \geq C(n)$$

   holds.

3. Let $M(n) = \{-1, -2, \ldots, -n\}$. For each nonempty subset we calculate the product of its elements. What is the sum of all these products?

4. Let $n > 4$ be an integer.

   We are given a convex n-gon $A_0A_1A_2...A_{n-1}A_n$ (with $A_n = A_0$) that is inscribed into a circle, and whose side lengths are given as $A_{i-1}A_i = i$ $(1 \leq i \leq n)$.

   Furthermore let φ_i be the angle between the line A_iA_{i+1} and the tangent to the circle in the point A_i. (Note: The angle between two lines is always smaller or equal yo $90°$.)

   Determine the sum

$$\Phi = \sum_{i=0}^{n-1} \varphi_i$$

   of these n angles.

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Part 2, Day 1, June 5, 2007

1. For which non-negative integers $a < 2007$ does the congruence $x^2 + a \equiv 0 \pmod{2007}$ have exactly two different non-negative solutions smaller than 2007?

   (I.e., there exist exactly two non-negative integers u and v smaller than 2007 such that $u^2 + a$ and $v^2 + a$ are divisible by 2007.)

2. Determine all sextuples $(x_1, x_2, \ldots, x_6)$ of non-negative integers that satisfy the following system of equations.

$$\begin{aligned} x_1x_2(1-x_3) &= x_4x_5 \\ x_2x_3(1-x_4) &= x_5x_6 \\ x_3x_4(1-x_5) &= x_6x_1 \\ x_4x_5(1-x_6) &= x_1x_2 \\ x_5x_6(1-x_1) &= x_2x_3 \\ x_6x_1(1-x_2) &= x_3x_4 \end{aligned}$$

3. Determine all rhombi (diamonds) $ABCD$ with sides of length $2a$ by giving their angle $\alpha = \measuredangle BAD$, such that the following holds:

   There exists a circle that intersects each side of the rhombus in a line segment of length a.

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Part 2, Day 2, June 6, 2007

4. Let M be the set of all polynomials $P(x)$ whose roots are pairwise distinct integers, and which have the property that the absolute values of all coefficients smaller than 2007.

   What is the highest degree among the polynomials in M?

5. We are given a convex n-gon with a triangulation, that is, a division into triangles by non-intersecting diagonals.

   Show that the n vertices can be marked with the digits of 2007 in such a way that each quadrilateral consisting of 2 neighbouring (along an edge) triangles has 9 as the sum of the digits at its 4 vertices.

6. We are given a triangle ABC with circumcircle k has midpoint U and radius r. On the extension of the radius UA a point P is chosen. The line PB is reflected about the line BA. The reflection is called g.

   Likewise, the line PC is reflected about CA. The reflection is called h.

   The intersection of g and h is called Q.

   Determine the set of all points Q when P varies on the entire extension of UA beyond A (that is, the points on the ray UA outside of the circle k).