



37th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 21, 2006

1. Let n be a non-negative integer whose decimal expansion ends with exactly k zeros but that is larger than 10^k .

For a certain n , only $k = k(n) \geq 2$ is known. How many possibilities (as a function of $k = k(n)$) are there at least to represent n as the difference of the squares of two non-negative integers?

2. Show that the sequence $\left\langle \frac{(n+1)^n n^{2-n}}{7n^2+1} \right\rangle_{n=0,1,2,\dots}$ is strictly increasing.
3. In a triangle ABC let D and E be points where the incircle touches the sides BC and AC . Show that if $\overline{AD} = \overline{BE}$, then the triangle is isosceles.
4. We are given the function $f(x) = \lfloor x^2 \rfloor + \{x\}$ that is defined for all positive real numbers. (Here for a real number u , $k = \lfloor u \rfloor$ is the largest integer smaller or equal u and $\{u\} = u - \lfloor u \rfloor$.) Show that there exists an infinite arithmetic sequence of distinct positive rational numbers that all have 3 as denominator when fully reduced and are not in the image of the function f .



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Part 2, Day 1, May 31, 2006

1. Let N be a positive integer.

Determine the number of positive integers $n \leq N$ that have an integer multiple whose decimal expansion only contains the digits 2 and 6 (not necessarily equally often).

2. Show that for all triple (a, b, c) of positive real numbers the inequality

$$3(a + b + c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}}$$

holds.

When does equality hold?

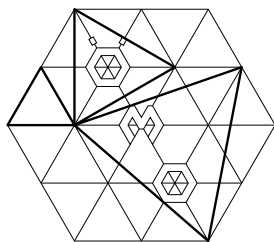
3. We are given a triangle ABC .

On the extension of AB beyond B we construct a point R with $\overline{BR} = \overline{BC}$ and on the extension of AC beyond C we construct a point S with $\overline{CS} = \overline{CB}$. Let A' be the intersection of the diagonals in the quadrilateral $BRSC$.

Analogously a point T with $\overline{CT} = \overline{CA}$ is constructed on the extension of BC beyond C , and a point U with $\overline{AU} = \overline{AC}$ on the extension of BA beyond A . Let B' be the intersection of the diagonals in the quadrilateral $CTUA$.

Furthermore a point V with $\overline{AV} = \overline{AB}$ is constructed on the extension of CA beyond A , and a point W with $\overline{BW} = \overline{BA}$ on the extension of CB beyond B . Let C' be the intersection of the diagonals in the quadrilateral $AVWB$.

Show that the area $F(AC'BA'CB')$ of the hexagon equals the sum of the areas $F(ABC)$ and $F(A'B'C')$ of the two triangles whose sides are the diagonals of the hexagon.



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Part 2, Day 2, June 1, 2006

4. Determine all rational numbers x such that $1 + 105 \cdot 2^x$ is the square of a rational number.
5. Find all monotonic functions $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(-f(x)) = f(f(x)) = f(x)^2$.
($f(x)$ is called monotonic if either $a < b \Rightarrow f(a) \leq f(b)$ or $a < b \Rightarrow f(a) \geq f(b)$.)
6. For integers $A \neq 0$ solve the following system of equations over $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

$$\begin{aligned}x + y^2 + z^3 &= A \\ \frac{1}{x} + \frac{1}{y^2} + \frac{1}{z^3} &= \frac{1}{A} \\ x \cdot y^2 \cdot z^3 &= A^2\end{aligned}$$