



36th Austrian Mathematical Olympiad  
Federal Competition for Advanced Students  
Part 1, May 30, 2005

1. Show that there exist infinitely many multiples of 2005 that contain all 10 digits  $0, 1, 2, 3, \dots, 9$  equally often (without counting leading zeros).
2. For how many integer values  $a$  with  $|a| \leq 2005$  does the system of equations

$$\begin{aligned}x^2 &= y + a \\ y^2 &= x + a\end{aligned}$$

have integer solutions?

3. For three real numbers  $a, b, c$  let  $s_n = a^n + b^n + c^n$  be the sum of their  $n$ -th powers. It is known that  $s_1 = 2$ ,  $s_2 = 6$  and  $s_3 = 14$ . Show that for all integers  $n > 1$  the identity  $|s_n^2 - s_{n-1}s_{n+1}| = 8$  holds.
4. We are given two congruent equilateral triangles  $ABC$  and  $PQR$  with parallel sides. The one is "pointing up", the other one "pointing down". The intersection of the triangle areas is a hexagon. Show that the three diagonals of the hexagon that connect opposite vertices pass through one common point.

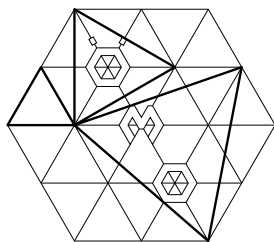


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Part 2, Day 1, June 8, 2005

1. Determine all triples  $(a, b, c)$  of positive integers such that  $a + b + c$  is the least common multiple of these three numbers.
2. Let  $a, b, c, d$  be positive real numbers.  
 Prove the inequality

$$\frac{a + b + c + d}{abcd} \leq \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}.$$

3. In an acute-angled triangle  $ABC$  two circles  $k_1$  and  $k_2$  are drawn whose diameters are the sides  $AC$  and  $BC$ . Let  $E$  be the foot of the altitude  $h_b$  on  $AC$  and let  $F$  be the foot of the altitude  $h_a$  on  $BC$ .  
 Let  $L$  and  $N$  be the intersections of the line  $BE$  with the circle  $k_1$  ( $L$  on the line  $BE$ ) and let  $K$  and  $M$  be the intersections of the line  $AF$  with the circle  $k_2$  ( $K$  on the line  $AF$ ).  
 Show that  $KLMN$  is a cyclic quadrilateral.



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4. The function  $f$  is defined on the set  $\{0, 1, 2, \dots, 2005\}$  assuming non-negative integer values, and satisfies the following three conditions for all non-negative integers  $x$  for which the arguments lie in the given set:

$$f(2x + 1) = f(2x), f(3x + 1) = f(3x), f(5x + 1) = f(5x)$$

How many different values can be assumed by the function?

5. Determine all sextuples  $(a, b, c, d, e, f)$  of real numbers that satisfy the following system of equations:

$$4a = (b + c + d + e)^4$$

$$4b = (c + d + e + f)^4$$

$$4c = (d + e + f + a)^4$$

$$4d = (e + f + a + b)^4$$

$$4e = (f + a + b + c)^4$$

$$4f = (a + b + c + d)^4$$

6. Let  $Q$  be a point inside a cube.  
Show that there are infinitely many lines  $g$  passing through  $Q$  such that  $Q$  is the midpoint of the line segment  $PR$  of  $g$  that lies inside the cube.