# 36th Austrian Mathematical Olympiad

## Federal Competition for Advanced Students

Part 1, May 30, 2005

1. Show that there exist infinitely many multiples of 2005 that contain all 10 digits $0, 1, 2, 3, \ldots, 9$ equally often (without counting leading zeros).

2. For how many integer values $a$ with $|a| \leq 2005$ does the system of equations

$$x^2 = y + a$$
$$y^2 = x + a$$

have integer solutions?

3. For three real numbers $a, b, c$ let $s_n = a^n + b^n + c^n$ be the sum of their $n$-th powers.
It is known that $s_1 = 2$, $s_2 = 6$ and $s_3 = 14$.
Show that for all integers $n > 1$ the identity $|{s_n}^2 - s_{n-1}s_{n+1}| = 8$ holds.

4. We are given two congruent equilateral triangles $ABC$ and $PQR$ with parallel sides. The one is ”‘pointing up”’, the other one ”‘pointing down”’. The intersection of the triangle areas is a hexagon.
Show that the three diagonals of the hexagon that connect opposite vertices pass through one common point.

© Dr. Gerd Baron. No responsibility is taken for the correctness of the details provided.

# 36th Austrian Mathematical Olympiad

## Federal Competition for Advanced Students

Part 2, Day 1, June 8, 2005

1. Determine all triples $(a, b, c)$ of positve integers such that $a+b+c$ is the least common multiple of these three numbers.

2. Let $a, b, c, d$ be positive real numbers.
   Prove the inequality
   $$\frac{a+b+c+d}{abcd} \leq \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}.$$

3. In an acute-angled triangle $ABC$ two cicles $k_1$ and $k_2$ are drawn whose diameters are the sides $AC$ and $BC$. Let $E$ be the foot of the altitude $h_b$ on $AC$ and let $F$ be the foot of the altitude $h_a$ on $BC$.
   Let $L$ and $N$ be the intersections of the line $BE$ with the circle $k_1$ ($L$ on the line $BE$) and let $K$ and $M$ be the intersections of the line $AF$ with the circle $k_2$ ($K$ on the line $AF$).
   Show that $KLMN$ is a cyclic quadrilateral.

© Dr. Gerd Baron. No responsibility is taken for the correctness of the details provided.

# 36th Austrian Mathematical Olympiad

## Federal Competition for Advanced Students

Part 2, Day 2, June 9, 2005

4. The function $f$ is defined on the set $\{0, 1, 2, \ldots, 2005\}$ assuming non-negative integer values, and satisfies the following three conditions for all non-negative integers $x$ for which the arguments lie in the given set:

$$f(2x+1) = f(2x), f(3x+1) = f(3x), f(5x+1) = f(5x)$$

How many different values can be assumed by the function?

5. Determine all sextuples $(a, b, c, d, e, f)$ of real numbers that satisfy the following system of equations:

$$\begin{aligned} 4a &= (b+c+d+e)^4 \\ 4b &= (c+d+e+f)^4 \\ 4c &= (d+e+f+a)^4 \\ 4d &= (e+f+a+b)^4 \\ 4e &= (f+a+b+c)^4 \\ 4f &= (a+b+c+d)^4 \end{aligned}$$

6. Let $Q$ be a point inside a cube.
Show that there are infinitely many lines $g$ passing through $Q$ such that $Q$ is the midpoint of the line segment $PR$ of $g$ that lies inside the cube.

© Dr. Gerd Baron. No responsibility is taken for the correctness of the details provided.