34th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 28, 2003

1. Determine all triples (p, q, r) of primes such that $p^q + p^r$ is a square number.

2. Determine the smallest and largest possible value of $f(x, y) = y - 2x$ for all non-negative real numbers $x,\ y$ with $x \neq y$ and $\frac{x^2+y^2}{x+y} \leq 4$.

3. Let t be a positive real number. Determine the number of positive real solutions (a, b, c, d) of the following system of equations.

$$\begin{aligned} a(1-b^2) &= t \\ b(1-c^2) &= t \\ c(1-d^2) &= t \\ d(1-a^2) &= t \end{aligned}$$

4. In a parallelogram $ABCD$, let E be the midpoint of the side AB and F the midpoint of BC. Let P be the intersection point of the lines EC and FD.

   Show that the segments AP, BP, CP and DP divide the parallelogram into four triangles with areas in $1 : 2 : 3 : 4$ ratio.

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Part 2, Day 1, June 28, 2003

1. Let the polynomial $P(n) = n^3 - n^2 - 5n + 2$ be given.

   Determine all integers n such that $P(n)^2$ is the square of a prime.

2. Let a, b, c be real numbers different from zero such that there exist $\alpha, \beta, \gamma \in \{-1, 1\}$ with $\alpha a + \beta b + \gamma c = 0$

   What is the smallest possible value of $\left(\frac{a^3+b^3+c^3}{abc}\right)^2$.

3. Around each grid point (x, y) with non-negative integers coordinates a square with the grid point as its center and length of its sides equal to $\frac{0{,}9}{2^x 5^y}$ is drawn in arbitrary orientation.

   Determine the area of this figure consisting of an infinit number of squares.

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Part 2, Day 2, June 29, 2003

4. Show that for each base $g > 2$ the exists exactly one three-digit number $(abc)_g$, that is represented as $(cba)_h$ with the digits in reverse order in a base h that differs from g by 1.

5. We are given a sufficient amount of bricks: Rectangles of the size 2×1 and squares of the size 1×1.

   Let $n > 3$ be a natural number.

   How many possibilities exist to fill a $3 \times n$ rectangle with these bricks in such a way that all 2×1 rectangles have their longer sides parallel to the side of length 3 and do not touch each other?

6. Let ABC be an acute-angled triangle. The circle k with diameter AB intersects the lines AC and BC in the points P and Q. Let R be the intersection point of the tangents in A and Q and let S be the intersection point of the tangents in B and P.

   Show that C lies on the line RS.